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Asymptotic approximation of the Shannon wavelet transform for large value of dilation parameter

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Abstract
In this paper we obtain asymptotic approximation of the Shannon wavelet transform for large value of dilation parameter when s⟶ + ∞, by using the previous results of extension of asymptotic approximations of the continuous wavelet transform.

Keywords: asymptotic expansion, wavelet transform, mellin convolution, integral transform, Shannon wavelet transform, dilation

1. Introduction

Keeping in mind the utility and interest in mathematics, physics and engineering, the concept of Shannon wavelet analysis and theory has been introduced by so many different authors. Signal analysis by ideal bandpass filters defines a decomposition known as Shannon wavelets. In functional analysis, a Shannon wavelet may be classified as a real or complex form. In particular, Shannon wavelets which are obtained from the real parts of the harmonic wavelets are a family of real functions. The analytical expression of the real Shannon wavelet can be found by taking the inverse Fourier transform. Shannon wavelets are studied together with their differential properties known as connection coefficients. Shannon scaling function is the starting point for the definition of the Shannon wavelet family. It is shown that the Shannon sampling theorem can be considered in a more general approach suitable for analyzing functions ranging in multifrequency bands. The approximation can be simply performed and the reconstruction by Shannon wavelets range in multifrequency bands [7].

The continuous wavelet transform of a function $f \in L^2(R)$ with respect to the wavelet $\emptyset \in L^2(R)$ is defined by [1]

$$(W_\emptyset f)(t,s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(u)\, \overline{\emptyset\left(\frac{u-t}{s}\right)}\, du,\ s > 0, t \in R, \tag{1}$$

provided the integral exists. The asymptotic expansion for Mellin convolution

$$I(\upsilon) = \int_0^\infty f(u) g(\upsilon u) du\ ,\ \text{as}\ \upsilon \to 0^+, \tag{2}$$

Let us rewrite (1) in the form [1]:

$$(W_\emptyset f)(t,s) = \vartheta^{\frac{1}{2}} \int_{-\infty}^{\infty} f(u+t)\, \overline{\emptyset(\vartheta u)}\, du$$

$$= \vartheta^{\frac{1}{2}} \left\{ \int_0^\infty f(u+t)\, \overline{\emptyset(\vartheta u)}\, du + \int_0^\infty f(-u+t)\, \overline{\emptyset(-\vartheta u)}\, du \right\} \tag{3}$$

$$= (W_\emptyset^{\ +} f)(t, s) + (W_\emptyset^{\ -} f)(t, s), \tag{4}$$

Where, $\vartheta = \frac{1}{s}$ and t is assumed to be a fixed real number. Setting $f(u+t) = \chi(u)$ and assume that $\chi(u)$ and $\overline{\emptyset(u)}$ are locally integrable on $(0, \infty)$. Further assume that asymptotic approximations of the form [1]:

$$\overline{\emptyset(u)} = \sum_{i=0}^{n-1} s_i\, u^{i-p} + \overline{\emptyset_n(u)},\ \text{as}\ u \to 0^+, \tag{5}$$

$$\chi(u) = \sum_{i=0}^{n-1} t_i\, u^{-i-q} + \overline{\chi_n(u)},\ \text{as}\ u \to +\infty. \tag{6}$$

Also assume that

$$\overline{\emptyset(u)} = O(u^{-\tau}),\ \text{as}\ u \to +\infty,\ \tau \in R, \tag{7}$$

$$\text{and}\ \chi(u) = O(u^{-\rho}),\ \text{as}\ u \to 0^+,\ \rho \in R. \tag{8}$$

With parameters $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\tau}$ and $\boldsymbol{\rho}$ satisfying the following condition:

$$\boldsymbol{p}+\boldsymbol{\tau}<1<\boldsymbol{q}+\boldsymbol{\rho},\ \boldsymbol{\tau}<\boldsymbol{q} \text{ and } \boldsymbol{p}<\boldsymbol{\rho}. \tag{9}$$

Let us remind earlier results [(8), (9), (10), of Theorem 1 [1]. The asymptotic approximation of continuous wavelet transform $(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s})$ given by (4) is given below by the following three cases of results [6] as:

Case I: When $\boldsymbol{n}=\mathbf{1},\mathbf{2},\mathbf{3}, \ldots$ and $\boldsymbol{m}=\boldsymbol{n}+\lfloor\boldsymbol{p}+\boldsymbol{q}\rfloor$ with $+\boldsymbol{q} \notin \boldsymbol{Z},$ we have

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{n-1} \boldsymbol{t}_i \left[\boldsymbol{M}\left[\overline{\emptyset(\boldsymbol{u})}\,; \mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right] + (-\mathbf{1})^{-i-q}\, \boldsymbol{M}\left[\overline{\emptyset(-\boldsymbol{u})}\,; \mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right] \right]$$
$$\times\ \boldsymbol{s}^{-i-q+\frac{1}{2}} + \sum_{i=0}^{n-1} \boldsymbol{s}_i\ [\boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u}); \mathbf{1}+\boldsymbol{i}-\boldsymbol{p}] + (-\mathbf{1})^{-i-p}\, \boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u}); \mathbf{1}+\boldsymbol{i}-\boldsymbol{p}]]$$
$$\times\ \boldsymbol{s}^{-i+p-1/2} + \mathbf{O}\,(\boldsymbol{s}^{-n-q+1/2}). \tag{10}$$

Case II: When $\boldsymbol{n}=\mathbf{1},\mathbf{2},\mathbf{3}, \ldots$ and $\boldsymbol{m}=\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{q}-\mathbf{1}$ with $\boldsymbol{p}+\boldsymbol{q} \in \boldsymbol{N},$ we have

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{p+q-2} \boldsymbol{s}_i\ [\, \boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u})\,; \mathbf{1}+\boldsymbol{i}-\boldsymbol{p}] + (-\mathbf{1})^{i-p}\, \boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u})\,; \mathbf{1}+\boldsymbol{i}-\boldsymbol{p}]]$$
$$\times\ \boldsymbol{s}^{-i+p-1/2} + \sum_{i=0}^{n-1} \boldsymbol{s}^{-i-q+1/2} \left\{ \lim_{z\to 0} \left[\boldsymbol{t}_i\, \boldsymbol{M}\left[\,\overline{\emptyset(\boldsymbol{u})}\,; \boldsymbol{z}+\mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right]\right.\right.$$
$$+ (-\mathbf{1})^{-i-q}\, \boldsymbol{M}\left[\overline{\emptyset(-\boldsymbol{u})}\,; \boldsymbol{z}+\mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right]$$
$$+ \boldsymbol{s}_{i+p+q-1}\ [\, \boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u}); \boldsymbol{z}+\boldsymbol{i}+\boldsymbol{q}\,]$$
$$\left.\left. + (-\mathbf{1})^{i+q-1}\, \boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u}); \boldsymbol{z}+\boldsymbol{i}+\boldsymbol{q}\,]\,]\,]\right\}\right.$$
$$+ \mathbf{O}\,(\boldsymbol{s}^{-m+p-1/2}\ \log(\mathbf{1}\backslash \boldsymbol{s})). \tag{11}$$

Case III: When $\boldsymbol{m}=\mathbf{1},\mathbf{2},\mathbf{3}, \ldots$ and $\boldsymbol{n}=\boldsymbol{m}+\mathbf{1}-\boldsymbol{p}-\boldsymbol{q}$ with $\mathbf{1}-\boldsymbol{p}-\boldsymbol{q} \in \boldsymbol{N},$ we have

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{-p-q} \boldsymbol{t}_i\, [\, \boldsymbol{M}\left[\,\overline{\emptyset(\boldsymbol{u})}\,;\ \mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right] + (-\mathbf{1})^{-i-q}\, \boldsymbol{M}\left[\,\overline{\emptyset(-\boldsymbol{u})}\,;\ \mathbf{1}-\boldsymbol{i}-\boldsymbol{q}\right]]$$
$$\times\ \boldsymbol{s}^{-i-q+1/2} + \sum_{i=0}^{m-1} \boldsymbol{s}^{-i+p-1/2} \left\{ \lim_{z\to 0} \left[\, \boldsymbol{t}_{i+1-p-q}\, [\boldsymbol{M}\left[\,\overline{\emptyset(\boldsymbol{u})}\,; \boldsymbol{z}+\boldsymbol{p}-\boldsymbol{i}\right]\right.\right.$$
$$+ (-\mathbf{1})^{-i-1+p}\, \boldsymbol{M}\left[\,\overline{\emptyset(-\boldsymbol{u})}\,; \boldsymbol{z}+\boldsymbol{p}-\boldsymbol{i}\right]\,]$$
$$+ \boldsymbol{s}_i\, [\, \boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u})\,; \boldsymbol{z}+\mathbf{1}+\boldsymbol{i}-\boldsymbol{p}]$$
$$\left.\left. + (-\mathbf{1})^{i-p}\, \boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u})\,; \boldsymbol{z}+\mathbf{1}+\boldsymbol{i}-\boldsymbol{p}]\,]\,]\right\}\right.$$
$$+ \mathbf{O}\,(\boldsymbol{s}^{-m+p-1/2}\ \log(\mathbf{1}\backslash \boldsymbol{s})). \tag{12}$$

2. Application

In this next section, we obtain asymptotic approximation of the Shannon wavelet transform by using aforesaid technique (10), (11) and (12), when $\mathbf{s} \to +\infty$ **.**

2.1 Asymptotic approximation of the Shannon wavelet transform

Let us consider $\emptyset$ to be a Shannon wavelet and it is given by [13] $\emptyset(\boldsymbol{u}) = \frac{\sin\left(\frac{\pi u}{2}\right)}{\left(\frac{\pi u}{2}\right)} \cos\left(\frac{3\pi u}{2}\right)$**.** Since, $\emptyset$ is locally integrable on $(\mathbf{0}, \infty)$ and has the asymptotic approximation [1]:

$$\overline{\emptyset(\boldsymbol{u})} = \mathbf{1} - \frac{7\,\pi^2 u^2}{6} + \frac{31\,\pi^4 u^4}{120} - \frac{217\,\pi^6 u^6}{5040} + \boldsymbol{O}(\boldsymbol{u}^7);\ \text{as } \boldsymbol{u} \to \mathbf{0}^+. \tag{13}$$

With $\overline{\emptyset(\boldsymbol{u})} = O(1)$; as $\to +\infty$ **.** (14)

Consider, $\boldsymbol{\chi}(\boldsymbol{u})$ is locally integrable on $(\mathbf{0}, \infty)$ and satisfy (6) and (8) with parameters

$$\mathbf{1} < q+\rho;\ \boldsymbol{q} > 0 \text{ and } \mathbf{0} < \boldsymbol{\rho}. \tag{15}$$

Now by using above results (10), (11) and (12) respectively and by means of formula ([7], p.321, (41)), then the asymptotic approximation of the Shannon wavelet transform for large value of dilation parameter when $\mathbf{s} \to +\infty$ are given below as:

Case I: When $\boldsymbol{m = 7 + \lfloor q \rfloor}$ and $\boldsymbol{q \notin Z}$, we get

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{6} \boldsymbol{t}_i\,(\mathbf{1} + (-\mathbf{1})^{-i-q})\,\{\,(\mathbf{1} - \mathbf{2}^{i+q})\,\boldsymbol{\pi}^{-1+i+q}$$
$$\times\ \mathbf{cos}\left[\frac{1}{2}\boldsymbol{\pi}(-\mathbf{1} + \boldsymbol{i} + \boldsymbol{q})\right]\boldsymbol{\Gamma}(-\boldsymbol{i} - \boldsymbol{q})\,\}\,\boldsymbol{s}^{-i-q+\frac{1}{2}} + \sum_{i=0}^{m-1} \boldsymbol{s}_i\,[\,\boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u}); \mathbf{1} + \boldsymbol{i}]$$
$$+ (-\mathbf{1})^{i}\,\boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u}); \mathbf{1} + \boldsymbol{i}]\,]\ \boldsymbol{s}^{-i-1/2} + \boldsymbol{O}\left(\boldsymbol{s}^{-\frac{13}{2}-q}\right). \tag{16}$$

Case II: When $\boldsymbol{m = 6 + q}$ and $\boldsymbol{q \in N}$, we get

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{q-2} \boldsymbol{s}_i\,[\,\boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u})\,; \mathbf{1} + \boldsymbol{i}] + (-\mathbf{1})^{i}\,\boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u})\,; \mathbf{1} + \boldsymbol{i}]\,]\ \boldsymbol{s}^{-i-1/2}$$
$$+ \sum_{i=0}^{6} \boldsymbol{s}^{-i-q+1/2}\,\{\lim_{z\to 0}\,[\boldsymbol{t}_i(\mathbf{1} + (-\mathbf{1})^{-i-q})\,\{\,(\,\mathbf{1} - \mathbf{2}^{i+q-z}\,)\,(\boldsymbol{\pi})^{-1+i+q-z}$$
$$\times \cos\left[\frac{1}{2}\,\boldsymbol{\pi}(-\mathbf{1} + \boldsymbol{i} + \boldsymbol{q} - \boldsymbol{z})\right]\boldsymbol{\Gamma}(-\boldsymbol{i} - \boldsymbol{q} + \boldsymbol{z}) + \boldsymbol{s}_{i+q-1}\,[\,\boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u}); \boldsymbol{z} + \boldsymbol{i} + \boldsymbol{q}\,]$$
$$+ (-\mathbf{1})^{i+q-1}\,\boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u}); \boldsymbol{z} + \boldsymbol{i} + \boldsymbol{q}\,]\,]\,]\,\} + \mathbf{O}\,(\boldsymbol{s}^{-m-1/2}\ \mathbf{log}(\mathbf{1}\backslash \boldsymbol{s})). \tag{17}$$

Case III: When $\boldsymbol{m = 6 + q}$ and $1 - \boldsymbol{q} \in \boldsymbol{N}$, we get

$$(\boldsymbol{W}_{\emptyset}\, \boldsymbol{f})\,(\boldsymbol{t}, \boldsymbol{s}) = \sum_{i=0}^{-q} \boldsymbol{t}_i\ \ (\mathbf{1} + (-\mathbf{1})^{-i-q}\,)\,\{\,(\,\mathbf{1} - \mathbf{2}^{i+q})\,(\boldsymbol{\pi})^{-1+i+q}$$
$$\times\ \mathbf{cos}\left[\frac{1}{2}\,\boldsymbol{\pi}(-\mathbf{1} + \boldsymbol{i} + \boldsymbol{q})\right]\,\boldsymbol{\Gamma}(-\boldsymbol{i} - \boldsymbol{q}\,)\,]\,\}\ \boldsymbol{s}^{-i-q+1/2} + \sum_{i=0}^{m-1}\ \boldsymbol{s}^{-i-1/2}$$
$$\times \{\lim_{z\to 0}\,[\,\boldsymbol{t}_{i+1-q}\,(\,\mathbf{1} + (-\mathbf{1})^{-i-1}\,)\,\{\,(\,\mathbf{1} - \mathbf{2}^{i+q-z}\,)\,(\boldsymbol{\pi})^{-1+i+q-z}$$
$$\times\,\mathbf{cos}\,\left[\frac{1}{2}\,\boldsymbol{\pi}(-\mathbf{1} + \boldsymbol{i} + \boldsymbol{q} - \boldsymbol{z})\right]\,\Gamma(-\boldsymbol{i} - \boldsymbol{q} + \boldsymbol{z})\,\}$$
$$+ \boldsymbol{s}_i\,[\boldsymbol{M}[\,\boldsymbol{\chi}(\boldsymbol{u})\,; \boldsymbol{z} + \mathbf{1} + \boldsymbol{i}] + (-\mathbf{1})^{i}\,\boldsymbol{M}[\,\boldsymbol{\chi}(-\boldsymbol{u})\,; \boldsymbol{z} + \mathbf{1} + \boldsymbol{i}\,]\,]\,]\,\}$$
$$+ \mathbf{O}\,(\,\boldsymbol{s}^{-m-1/2}\ \mathbf{log}(\mathbf{1}\backslash \boldsymbol{s}\,)\,). \tag{18}$$

3. Advantages of the Shannon wavelet

The Shannon wavelet is the most important tools to measure for analysis of impulse functions. The approximation can be simply performed and the reconstruction by Shannon wavelets range in mutlifrequency bands. Shannon sampling theorem plays a fundamental role in signal analysis and in particular for the reconstruction of a signal from digital sampling. It has been recognized that on the Sinc function one can settle the family of Shannon wavelets [10]. Shannon wavelet theory is based on a family of orthogonal functions having many interesting properties. Shannon wavelets are related to the harmonic wavelets, being the real part thereof and to the well known sinc function, which is the basic function in signal analysis. It should be also noticed that as compared with other wavelet families, the main advantages of Shannon wavelets is that they are analytical functions, thus being infinitely differentiable. As Shannon connection coefficients can be easily defined but their explicit values usually require lengthy computations. Only by using a computer algebra symbolic system it was possible to obtain a finite formula for calculating their numerical values. Infact, this formula is not evident a priori but it can be summarized only after the computation of a large amount of numerical sequence. The connection coefficients of Shannon wavelets are explicitly computed with a finite formula upto any order [12].

4. Conclusion

We can easily compute the approximation terms of Shannon wavelet transform with their exact error terms for large value of dilation parameter when $\mathbf{s} \to +\infty$ and get the above following results (16), (17) and (18) respectively.

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