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# Flexible algebraic action on quadratic equations

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**Abstract**

This paper describes a study that explores the competencies of flexible algebraic action of German students in grade nine and ten when dealing with quadratic equations. A theoretical framework for the concept of flexibility in algebraic action in the context of quadratic equations is provided. Further on, data analysis and some important early results of the study are discussed. The study examines which features of quadratic equations the students perceive, what meanings they infer from these features and to what extent this is conducive to or obstructive for flexible algebraic action. Two types of meta-tasks were used in the study and analyzed with qualitative data analysis methods.

**Keywords:** quadratic equations, flexibility, algebra

## Introduction

When professional mathematicians solve quadratic equations like (1) $x2 + x - 6 = 0$, (2) $x2 + 2x = 0$, (3) $(x - 3)(x + 5) = 0$ or (4) $(x - 3)(x + 5) = 7$ they will use different methods to find the solutions in effective and less error-prone ways. They do so, because they recognize different features of the equations and they are able to draw appropriate conclusions for solving the equations. E.g., equation (1) and (2) look very similar with a sum as the term on the left-hand side and zero on the right-hand side. The difference is the missing constant in (2), which indicates that this equation can be easily solved by factoring. In contrast, using the pq-formula[1] is a suitable procedure for solving equation (1). The equations (3) and (4) have the same structure on the left-hand side. The only, but important difference is the number on the right hand side. The solutions of (3) can be immediately determined without any calculation, whereas for solving (4), it is necessary to expand the brackets and use the pq-formula. Although all quadratic equations can be solved by using the pqformula, for (2) und (3) it is not an effective way because the calculations performed before or while using the pq-formula are not necessary. Furthermore these calculations are error-prone, especially for students with problems in algebraic conversions. When they use the pq-formula for (2) a common mistake is using a wrong value for q in the formula. The expanding of brackets in (3) is also a wellknown field of mistakes. Using different solution methods depending on the characteristics of the equation can be called flexibility in contrast to the use of only one standard routine like the pq-formula for each type of quadratic equations.

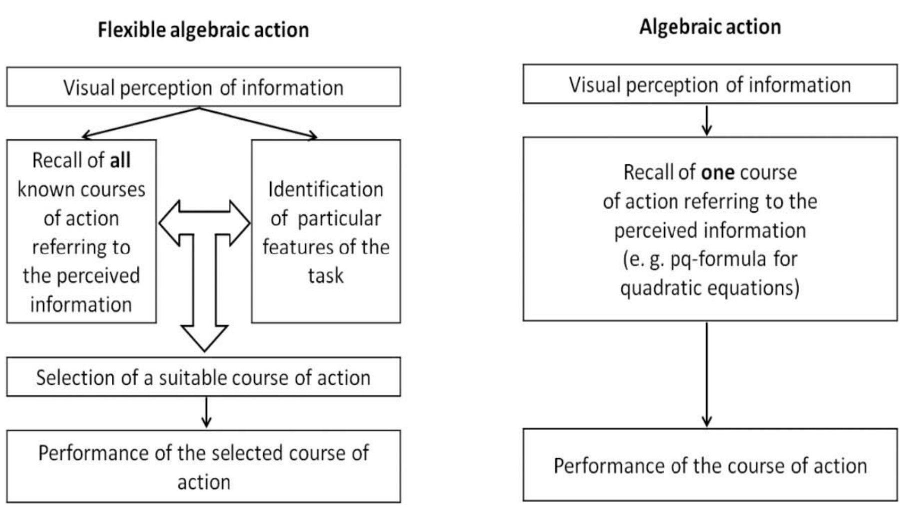


**Fig 1:** Comparison of flexible algebraic action and algebraic action

## Quadratic Equations and Flexible Algebraic Action

Flexible algebraic action is defined as the ability to choose an adequate processing method depending on the specific features of the task and the abilities of the individual. This definition refers to the concept of flexibility in mental calculation (e.g. Rathgeb-Schnierer, 2006; Threlfall, 2002) [10, 12]. And a general discussion about what flexibility can mean (e.g, Star & Newton, 2009) [14]. Figure 1 shows the comparison of flexible algebraic action and algebraic action just with one standard routine.

For quadratic equations a didactical map can show the complexity of the situation students have to cope with, when they learn to solve this type of equation. A didactical map is a graphic depiction on an issue which contains important information for didactical considerations under special questioning. To clarify the difference between linear and quadratic equations in situations of learning and regarding the necessity of flexibility, a didactical map of linear equations (Figure 2) will be contrasted to a didactical map of quadratic equations (Figure 3). The construction refers to the "Didactical cut" which was first named by Filloy and Rojano (1984, 1989) [4, 5] and later on discussed by several researchers (e.g., Herscovics & Linchevski, 1994; Lima & Healy, 2010; Vlassis, 2002) [3, 7, 16].

The map shows, that linear equations can be divided into two main groups: In the first, the unknown is only appearing once on one side of the equation. These equations can be solved by arithmetical procedures. It is not necessary to act on or with the unknown because they can be solved by using the reverse operations, e.g. $3x + 7 = 19$ can be solved by calculating $(19 - 7) \div 3$. To solve the second group of equations, in which the unknown occurs on both sides or more than once on one side, it is necessary to use algebraic procedures to act on or with the unknown. According to this classification, Lima and Healy (2010) [7] call these two groups 'evaluation' and 'manipulation' equations which resembles the classification by Filloy and Rojano (1984, 1989) [4, 5]. for linear equations, but which is farther-reaching also for classifying quadratic equations. Lima and Healy focus on the activities which are necessary to solve an equation and not on the question, where or how often the variable occurs. In contrast to the evaluation equations, for the manipulation equations it is necessary to manipulate algebraic symbols. The group of manipulation equations can be divided into two subgroups. For the first, where the variable is only appearing on one side but more than once, algebraic procedures are only necessary for the terms on one side. For the second, where the variable appears on both sides, equivalent transformations on both sides of the equation are necessary. To describe the important differences between the three groups regarding the different requirements of arithmetical and algebraic skills, the author suggests using the term "cognitive step" which is also suitable for the quadratic equations.

The aim of each algorithmic solving process for linear equations is to transform the equation into the form of the first group. The possible transformations are indicated by the arrows on the map. One facet of flexibility on acting with linear equations should be

e.g. the ability to recognize that $3x + 4 = 3x + 5$ has no solution without starting algebraic procedures on this equation. The map shows that the field of linear equations has got a manageable number of cases. Nevertheless, there are also flexible and intelligent strategies to solve linear equations by simplifying the given equation without strictly using algebraic

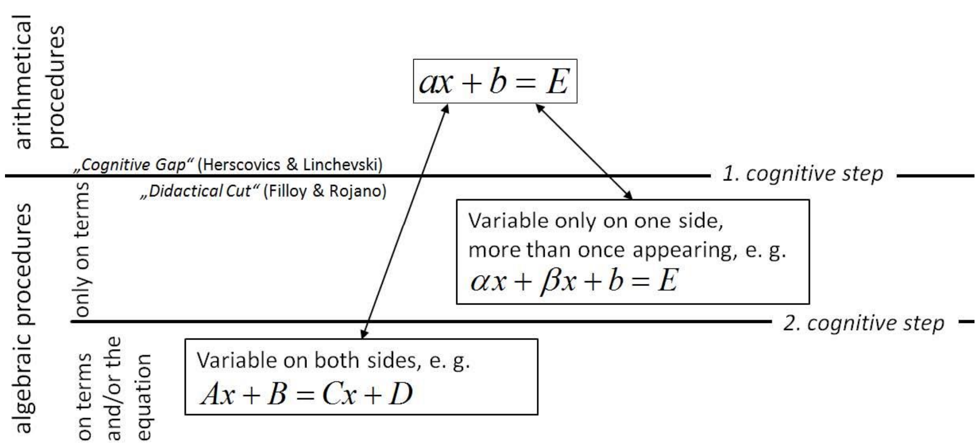


**Fig 2:** Didactical map of linear equations

algorithms (e.g., Star & Rittle-Johnson, 2008) [15]. For example for solving the equation $5(x + 2) = 20$ it is not necessary to expand the brackets, when recognizing that the term in brackets has to be 4 and then solving the equation $x + 2 = 4$ instead of the given equation. But this type of flexibility is depending only on special numbers. The idea behind this is to get an equation which is solvable with arithmetic procedures but the way to achieve this type depends on the numbers and the structure and it is not strictly algorithmically performing.

In contrast to this fairly simple model, the didactical map of quadratic equations (Figure 3) shows the wide variety of types of quadratic equations under the view of different effective solution methods. The study is only focussing on basic types of quadratic equations and not on non-standard examples like $\sin2(x) + 2\sin(x) + 1 = 0$ or $x4 - 6x2 + 9 = 0$, for which identifying and interpreting of features as a basis for acting flexible is also very important. There are two main groups of equations: The first is solvable with quasi-arithmetical procedures such as inverse operations, extracting radicals or using the fact, that a product equals zero if one of the factors equals zero. The last case is indicated as a special case by the vertical spotted line, because a special knowledge is needed and no arithmetic operations are necessary. To solve the second group, algebraic procedures are necessary. This group can be divided into two subgroups: The first is characterized by the fact that the algebraic procedures are explicitly done when factoring the equation with the missing constant term. After this, the solutions are obvious using the knowledge about cases when a product equals zero. Using the pq-formula for all other types, the algebraic procedure of solving is not completely visible, so it is called implicit. This classification of the two main groups is according to the terms 'evaluation' and 'manipulation' equations Lima (2007) [7] used. In contrast to the linear equations, the cognitive steps do not depend on the fact where and how often the variable appears.

The dashed arrows indicate that some equations can be interpreted as special cases of other types of equations. The types of equations differ in the types of the terms appearing. The structure of the terms is indicated by the form of the frames. A product is indicated by a frame shaped like an ellipse and a sum by a hexagon. The rectangle indicates the special case, where the unknown appears linear in a product but the term is a sum. To indicate the different suitable methods of solving, the frames have different kinds of lines. The dashed lines indicate the *pq*-formula, the dot-dash-lines extracting radicals as an appropriate method and the dot-dot-dash-lines stand for the possibility to get the solutions by factoring the term.

It is obvious that the types of quasi-arithmetical and algebraic solvable equations correspond in a complex way. For choosing an effective solution method it is not sufficient to look just at the type or structure of the term on one side of the equation. It is also necessary to look at the structure of the equation as a whole and the appearing numbers at special places in the equation (e.g. zero on one side of the equation). The main difference between the linear and the quadratic equations in the process of flexible solving is, that for quadratic equations it is not the aim to transform all types just into one, which can be solved by a stand

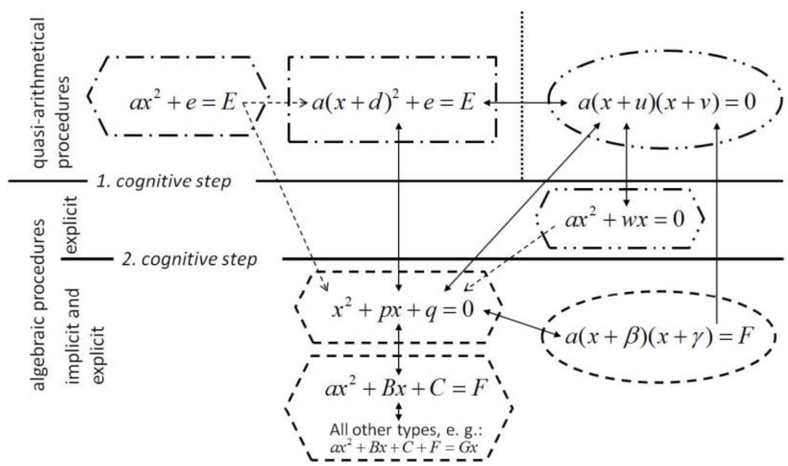


**Fig 3:** Didactical map of quadratic equations

ard method, as it is with the linear equations where it is the aim to produce a type of equation which can be solved by arithmetical procedures. Flexibility in solving quadratic equations means choosing different algorithmic solving methods, depending on special features of the equations. To master this, it is required to consider the relationships shown in the didactical map and to know, which features of an equation are important to indicate a suitable more or less algorithmic solving method.

**Method of the Study**

The three main questions in this study are

1. Which features of quadratic equations do students perceive?
2. What meanings do they infer from these features?
3. To what extent is this conducive to or obstructive for flexible algebraic action?

To answer these questions two types of studies were made: 1. Laboratory study in a one-to-one situation (researcher and participant) with eleven students from grade nine from four different classes from two different German high-schools (Gymnasium). 2. Classroom study with 26 students from grade nine and 20 students from grade 10 from yet another Gymnasium. In the classroom study the tasks were integrated in a lesson by a teacher who was exactly instructed how to moderate the lesson. In the German high-school curriculum normally quadratic equations are a topic at the end of grade eight, so that all participants had taken part in lessons about quadratic equations and regarding this the groups are comparable. From another point of view potential varieties of the two groups in the classroom study can be recognized during the process of data-analysis. The frame-data from the participants in the laboratory study, marks for mathematics and results of the DEMAT 9-Test (Schmidt, Ennemoser, & Krajewski, 2013) [13] reveal that this is a mixed group regarding the level of general mathematical skills. For the classroom study frame-data couldn't be collected. The teacher reported that both classes show no abnormality regarding the level of general mathematical skills.

In the laboratory study, the participants had to process three tasks. In the first task, a quadratic equation was given and the students were asked to create new equations by varying the given one. In the second task, they had to solve five quadratic equations of different types to check which strategies the students use to solve the equations.

The third task, which is at the centre of the study, is a meta-task like the first one. 20 quadratic equations were given on cards of carton and the participants had to assort them. These equations represent the different categories shown in the didactical map of quadratic equations, e.g. the equations discussed in the introduction are part of the selection. The number 20 was chosen based on the time needed to get an overview of the equations and the capacity of the visual field. There were no rules given and the participants had all freedom to assort the equations as they like. It was remarked that there was not only one possibility to assort them. While working on the tasks, the participants were video recorded and asked to think aloud. The students were asked to explain the meanings of the features of the groups.

The participants of the classroom study had to process two tasks. The variation-task was left out because the laboratory study showed that the most interesting information of this task was given by the thinking aloud of the participants which was not recordable in the classroom study. Deviating from the laboratory study, the students had to work in pairs on the sorting-task to encourage multiple solutions for the assorting. This seemed to be necessary because there was no researcher beneath the participants during process on the task like in the laboratory study to initiate more than one solution. The participants had to write down their assorting with an explanation of the meanings of the features on a special documentation sheet which referred to a tool used in sorting-tasks in a study with teachers by Zaslavsky and Leikin (2004) [19]. Selected pairs of students presented their arguments for assorting and their reasoning in the class which was recorded by video.

The transcriptions of the videos and the documentation sheets for the sorting-task were analyzed with qualitative data analysis methods, like an open coding, with the aim to develop categories (cf. Corbin & Strauss, 2008) [1] of reasoning for assorting and the meanings of the recognized features of the equations. The sorting-task as an analyzingmeta-task is particularly suitable to examine the questions of the study. To process on this task it is insufficient to know a routine to handle quadratic equations. It is necessary to have an explicit look on the features of the equations and to detect the syntactic or semantic differences which are preconditions for flexible algebraic action. By reasoning for assorting, the meanings of the features can be explained. Evaluating the identified categories of assorting and meanings can show how far the mental structures of the students are conducive to or obstructive for flexible action.

**Data analysis: selected results of the sorting-task**

Analyses of the data in the classroom studies reveal that there were six main categories for assorting the equations, which were also found in the laboratory study. These categories and some main sub-categories are shown in Figure 4. The meanings inferred from the features for assorting the equations were first and mainly evaluated by analyzing the interviews from the laboratory study because the videos contain much more information than the documentation sheets from the classroom study. The meanings can be divided into helpful and conducive to or obstructive for flexible algebraic action. There are also features and meanings which are obscure for flexible action but they show insight into the students' mental concept concerning the dealing with equations. In the following selected examples, features and reasons for assorting are discussed.

One dominating reason for the assorting in the category "Term" was the appearing of brackets in the equations. A lot of participants argued, that terms with brackets had to be expanded to simplify them. This result is in accordance with a result Lima and Tall (2006) [8] found in a study, where the quadratic equation $(x-3)\cdot(x-2)=0$ was (not successfully) solved by the most participants by expanding the brackets. Similar difficulties with equations in this structure were also noticed by Vaiyavutjamai, Ellerton and Clements (2005) [17]. The brackets operate as a signal to expand the term regardless whether it is necessary or useful or not. In other studies (e.g., Dreyfus & Hoch, 2004; Wenger, 1987) [2, 18], other signals like radicals or fractions, which evoke routines regardless of the context or the questions that should be answered, were identified. Focussing on such signals could be an indication, that the students do not plan their approach (cf. Wenger, 1987) [18] which is a necessary step in flexible action. Expanding the brackets can prevent an efficient solving process and can lead to mistakes during expansion or the subsequent use of the pq-formula. One reason for this habit can be the way transforming terms is taught. When transforming is called "simplifying" and most of the tasks require expanding products then it is obvious that for most students, brackets have to be expanded in any case. This hypothesis is supported by the explanation that terms with brackets are more complicated, which was remarked by a lot of participants of the laboratory study.

The remarks about brackets often occur together with remarks about solving methods. The participants describe that it is more difficult to isolate the variable when it occurs in brackets. This argument mirrors the strategy for linear equations, i.e. isolating the variable on one side of the equation, which was used in the solvingtask for 26.8% of the equations (from all samples) but successful only for 45.1% of these equations. The correct solutions with this strategy were all produced for the equation $(x-8)2=0$ by using the inverse operations or arguing with the semantics of

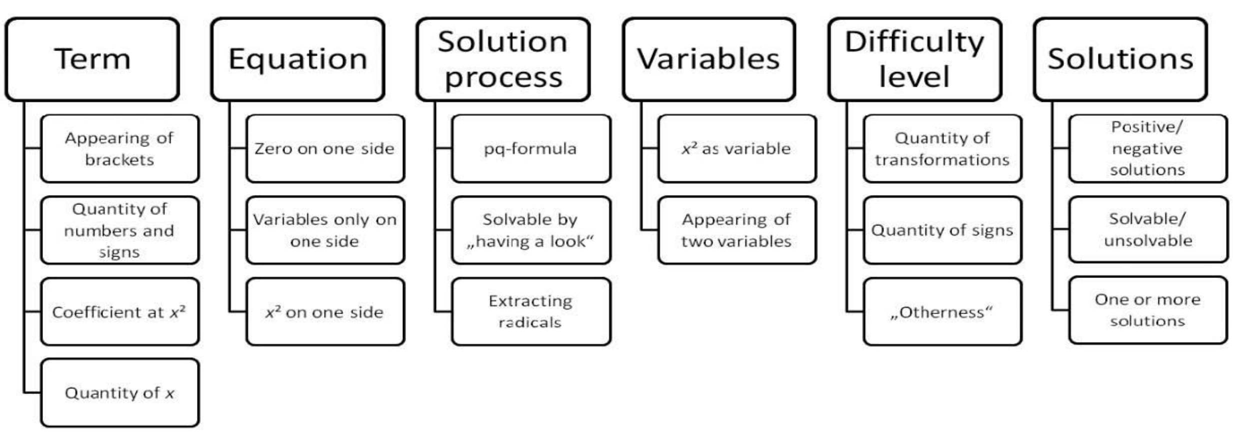


**Fig 4:** Categories and main sub-categories of assorting quadratic equations

This equation. The other strategies used for isolating the variable, misapplied on equations for which the pq-formula or factorizing is a suitable solving method, were division by the variable $x$ or eliminating the exponent of $x2$ by a division by 2, by extracting radicals just from this monomial or by summarizing monomials with different exponents. These faulty strategies of isolating the variable are all related to the idea of solving the equation by extracting the linear variable on one side of the equation to find the solution then on the other side. This is the idea of solving linear equations which works for quadratic equations only in special cases. Obviously, a generalization of this strategy leads to the effect that it is used in situations where it is unsuitable. The argumentation of the participants in the sortingtask points to a lower level of understanding of the concept of equations in general and especially of different types of equations like linear and quadratic equations. This hypothesis is promoted by results of studies about the understanding of equations (e.g., Lima & Tall, 2008) [9] The isolation-strategy, which evokes the expanding of brackets, is opposed to strategies for solving quadratic equations where brackets are produced when factoring a term or where the brackets indicate a simple way to find the solutions because they show that the term is a product. A more general problem is the mentioned difficulty to identify the equations as quadratic when brackets are appearing. This topic responds to the aspect of symbol sense in algebra in the context of understanding the meaning of variables and parameters in equations (cf. Postelnicu & Postelnicu, 2015) [11] and to the ability to anticipate the effect of transforming terms. Another feature that appeared in the category "Equation" for nearly every participant as a category was the presence of zero on one side of the equation. In contrast to the reasons of the importance for the occurrence of brackets, no blocking points for flexible action were found in the argumentation for this feature. However, only one pair of students in the classroom study explained, that the occurrence of zero on one side of the equation is the necessary precondition to use the pq-formula or to find the solutions when the term on the other side is a product. A lot of participants explained that zero on one side is necessary to use the pq-formula. This explanation is not wrong, but it was referred to all equations with zero on one side, regardless of the type and structure of the term on the other side of the equation, e.g. $(x - 3)(x + 5) = 0$ or $4x2 - 10 = 0$ which can be easily solved by the fact that a product should be zero or by extracting a radical. The students focussed only on one single feature not following the need to analyse potential sub-features of this group of equations. If this feature works as a signal to use the pq-formula, it can be obstructive for flexible algebraic action. This is as much more remarkable when looking on the results of the solving-task. From all samples 34.3% of the equations were solved by using the pq-formula, 70.3% of these correct. This indicates, that the pq-formula as a standard-method is not executed as well as you can expect for a standard algorithm. These results are compatible to a study by Lima and Tall (2006) [8] where most participants solved quadratic equations with trial-and-error or with the pq-formula, but mostly unsuccessfully. Similar to this focussing on only one single feature, for the feature $x2$on one side of the equation some participants argued that if $x2$ is on one side, extracting radicals is a suitable solving method without regarding what is on the other side of the equation, e.g. $x2 = x$ or $x2 = -16x - 64$. Following the faulty meaning of this feature and the revealing solving method can produce individual faulty strategies to handle the other side and wrong solutions.

Another explanation for assorting by zero on one side was that there are special rules to be regarded when a zero appears. It is true that the special rules (like division by zero is not possible) are valid for handling equations. This argumentation, at a first glance, seems not to be connected to flexibility therefore it is neither conducive to nor obstructive for flexible action. But if this is the only and dominating importance of this feature in the awareness of the students, it can overlay meanings which are important for flexible action and in this way it can be obstructive.

## Conclusion

The first data analyses of the sorting-task show some important findings regarding the competencies of flexible algebraic action in the context of quadratic equations. A lot of explanations in the sorting-task which can be obstructive for flexible action were identified and only a few participants show that their understanding of quadratic equations and solving methods is based on a concept of flexibility. The qualitative analyses used in this study are appropriate to identify the reasons for the established deficits. This knowledge can be used to develop improvements for teaching. The results show that some problems the students have with quadratic equations are founded in the teaching of previous topics (like transforming terms and the dominance of expanding brackets). Other problems like focussing just on one feature (e.g. zero on one side or $x2$ on one side) should be addressed in the lessons by using suitable types of tasks which focus not on finding solutions of equations but on classifying different types of equations. It would seem that the meta-tasks, used in this study, have the potential to be a useful tool for the design of mathematic lessons which aim to enable the learners to act flexibly.

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