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Harvesting policy and stability analysis of a food chain model

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Abstract
In this paper we consider a tri-tropic food chain model and calculate the condition of stability of the equilibrium points analytically. Numerically support has been given for this system to for better understanding of this analytical phenomenon. We have also try analytically investigated the impact of harvesting the prey and predator population at maximum sustainable yield policy level of the system for appropriate use of biological resources.

Keywords: routh-hertz criteria, food chain model, maximum sustainable yield policy, harvesting phenomenon

1. Introduction
One key role of mathematics in biology is the creation of mathematical models. These are equations or formulas that can predict or describe natural occurrences, such as organism behavior patterns or population changes over time. While formulating model, a biologist has to think very carefully about premises of the model. They have to be realistic. Use of biological resources in a sustainable way is one of the current research interests to fulfill the sufficient requirement of human needs in the long run. Due to over exploitation many biological resources are in danger of extinction and many of them are being depleted. To protect biodiversity and ecological balance many researchers are recommended maximum sustainable yield policy (MSY) and maximum economic yield policy. MSY policy was first introduced by Schaefer (1954) [1] in a long back ago for a single species fishery having logistic law of growth and subject to proportional harvesting. Clark (1990) [2] also discussed the importance of the concept of MSY policy in fishery management. Recently Kar [3] has observed the MSY and MSTY policy on food chain model. Kar and Matsuda (2007) [4] studied the impacts of MSY policy in a single species fishery. Chattopadhyay and Arino [5] studied predator-prey system when predator eat infected prey and derived the persistence and extinction conditions and also determined the condition for Hopf bifurcation. Xiao and Chen [6] modified the model of Chattopadhyay and Arino by introducing the delay term and studied the dynamics of the modified system. Mukherjee [7] analyzed a generalized prey-predator system with parasite infection and obtained conditions for persistence and impermanence. Roy and Chattopadhyay [8] introduced a mathematical model of disease-selective predation incorporating this concept. They considered a predator-prey system where the predator has specific choice regarding predation and it can recognize the infected prey and avoid those during predation. Holmes and Bethel [9] discussed many examples in which the parasite changes the external features or behavior of the prey, so that infected prey are more vulnerable to predator. Infected prey sometimes chooses such locations that are more accessible to predators; for example, infected fish or aquatic snails may live close to the water surface or snails may live on top of vegetation rather than under protective plant cover. Similarly, infected prey sometimes became weaker or less active, so that they are caught more easily by predator (see [10]). Fisheries management is often focuses on maximizing the yield of a single targeted species and ignores habitat, predator and prey of the target species and other ecosystem components interactions [12]. In this paper we consider a tri-tropic food chain model and first find out the different equilibrium points of the model system and its existence criterion. We also find out the condition of stability of the equilibrium points analytically and numerically support to the result for better understanding of this analytical phenomenon. On the second portion we have also analytically investigate the impact of harvesting the prey and predator population at maximum sustainable yield policy level of the model system.

2. Mathematical Model formation
In this section we assume certain things before formation the tri-tropic food chain model some assumption are considered. In this system x is the biomass of the prey population with carrying capacity k. We consider y and z as the biomass of predator (second tropic level) and top predator (third tropic level) respectively at any time t. Here b measures the strength of predation by the top predator on the predator y and γ is the natural death rate of the top predator. Also r is a constant intrinsic growth rate and k is the environmental carrying capacity of the prey species. Here a is the predation rate and m is the natural mortality rate of the predator. Under the following assumption we consider a tri-tropic food chain model as follows:

$$\frac{dx}{dt} = rx\,log\left(\frac{k}{x}\right) - axy$$

$$\frac{dy}{dt} = axy - bzy - my$$
$$\frac{dz}{dt} = bzy - \gamma z \qquad (1)$$

3. Qualitative Analysis of the Model System

3.1 Equilibrium points and it existence criterion

1. Both the predator free equilibrium point $E_1(k, 0, 0)$ always exists.
2. Top predator free equilibrium point $E_2\left(\frac{m}{a}, \frac{r}{a}\log\left(\frac{ka}{m}\right), 0\right)$ exists if $ak > m$.
3. The interior or coexisting equilibrium point $E_3\left(ke^{-\frac{a\gamma}{br}}, \frac{\gamma}{b}, \frac{1}{b}\left(ake^{-\frac{a\gamma}{br}} - m\right)\right)$. This equilibrium point exists if $k > \frac{m}{a}e^{\frac{a\gamma}{br}}$. It is also observe that existence of E_3 ensures the existence of E_2.

3.2 Stability Analysis

Lemma1. The planar equilibrium point $E_2\left(\frac{m}{a}, \frac{r}{a}\log\left(\frac{ka}{m}\right), 0\right)$ is locally asymptotically stable if $\frac{br}{a}log\left(\frac{ka}{m}\right) < \gamma$ and $rmlog\left(\frac{ka}{m}\right) < 0$.

Proof: The Jacobeans matrix of the above system at the equilibrium point $E_2\left(\frac{m}{a}, \frac{r}{a}\log\left(\frac{ka}{m}\right), 0\right)$ is

$$J(E_2) = \begin{pmatrix} \frac{rm^2}{a^2k} & -m & 0 \\ rlog\left(\frac{ka}{m}\right) & 0 & -\frac{br}{a}log\left(\frac{ka}{m}\right) \\ 0 & 0 & \frac{br}{a}log\left(\frac{ka}{m}\right) - \gamma \end{pmatrix}.$$

And its characteristic equation is $\begin{vmatrix} \frac{rm^2}{a^2k} - \lambda & -m & 0 \\ rlog\left(\frac{ka}{m}\right) & -\lambda & -\frac{br}{a}log\left(\frac{ka}{m}\right) \\ 0 & 0 & \frac{br}{a}log\left(\frac{ka}{m}\right) - \gamma - \lambda \end{vmatrix} = 0.$ one of the Eigen values is $\frac{br}{a}log\left(\frac{ka}{m}\right) - \gamma$ and other two

Value can be obtained from the relation $\lambda^2 - \lambda\frac{rm^2}{a^2k} + rmlog\left(\frac{ka}{m}\right) = 0$ and it eigen values are always negative if $rmlog\left(\frac{ka}{m}\right) < 0$. Hence the proof is completed.

Lemma 2. The interior equilibrium point $E_3\left(ke^{-\frac{a\gamma}{br}}, \frac{\gamma}{b}, \frac{1}{b}\left(ake^{-\frac{a\gamma}{br}} - m\right)\right)$ is asymptotically stable if, $\left(rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m)\right) < 0$, $\left(\frac{a}{b}(akp - m) - rkp^2 - \frac{a\gamma}{b}\right)\left(\frac{kp\gamma a^2}{b}\right) + (akp\gamma - m\gamma)\left(\left(rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m)\right)\right) > 0$ and $(m\gamma - akp\gamma)\left(\left(rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m)\right)\right) < 0$.

Proof: For this we first find out the Jacobean matrix. At the interior equilibrium point $E_3\left(ke^{-\frac{a\gamma}{br}}, \frac{\gamma}{b}, \frac{1}{b}\left(ake^{-\frac{a\gamma}{br}} - m\right)\right)$

The Jacobean matrix is $J(E_3) = \begin{pmatrix} rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m) & -akp & 0 \\ \frac{a\gamma}{b} & 0 & -\gamma \\ 0 & -akp - m & 0 \end{pmatrix}$ and the characteristic equation is

$$\begin{vmatrix} rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m) - \lambda & -akp & 0 \\ \frac{a\gamma}{b} & -\lambda & -\gamma \\ 0 & -akp - m & -\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - \lambda^2\left(rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m)\right) + \lambda\left(\frac{kp\gamma a^2}{b}\right) - (akp\gamma - m\gamma)\left(\left(rkp^2 + \frac{a\gamma}{b} - \frac{a}{b}(akp - m)\right)\right) = 0.$$

Using Routh-Hurwitz criterion, each of the three roots are with negative real part if

$$\left(rkp^2+\frac{a\gamma}{b}-\frac{a}{b}(akp-m)\right)<0,$$
$$\left(\frac{a}{b}(akp-m)-rkp^2-\frac{a\gamma}{b}\right)\left(\frac{kp\gamma a^2}{b}\right)+(akp\gamma-m\gamma)\left(\left(rkp^2+\frac{a\gamma}{b}-\frac{a}{b}(akp-m)\right)>0\right.$$ and
$$(m\gamma-akp\gamma)\left(\left(rkp^2+\frac{a\gamma}{b}-\frac{a}{b}(akp-m)\right)<0\right.$$. Hence the proof is complete.

4. Numerical Simulation

We now try to solve the model system by numerically to understand the analytic behavior of the model. We first consider the 2D plot of both the planar and interior equilibrium point and the 3D phase portrait of the corresponding equilibrium point. This is given below to support the analytical characteristic of the model system.

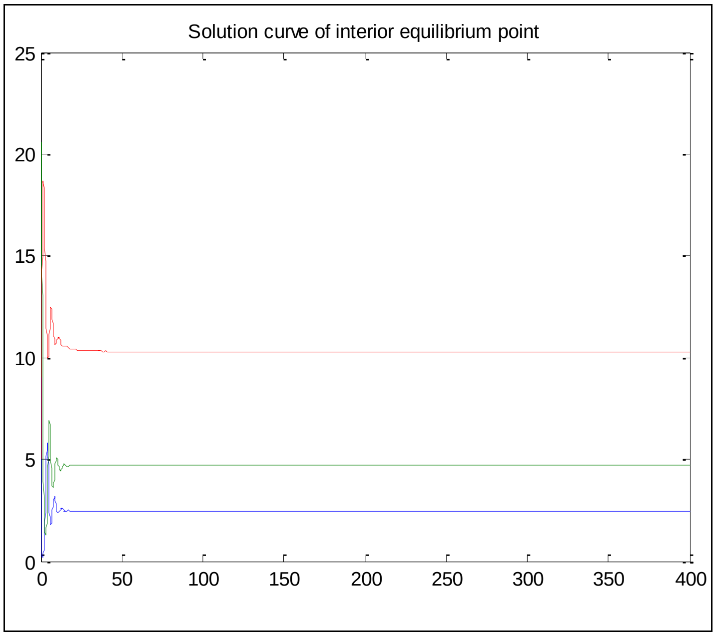

Fig 1: This is the stability of the interior equilibrium point of the model system under the parametric values as r=0.85; k =20; a=0.38; b=0.089; m=0.0048; γ=0.42.

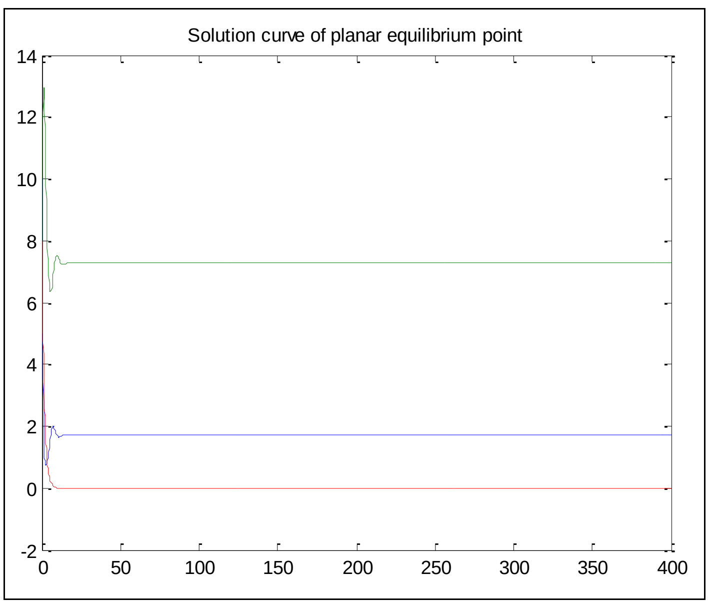

Fig 2: This is represent the stability of the planar equilibrium point of the model system taking the parametric values as r=0.71; k =20; a=0.24; b=0.010; m=0.41; γ=0.75.

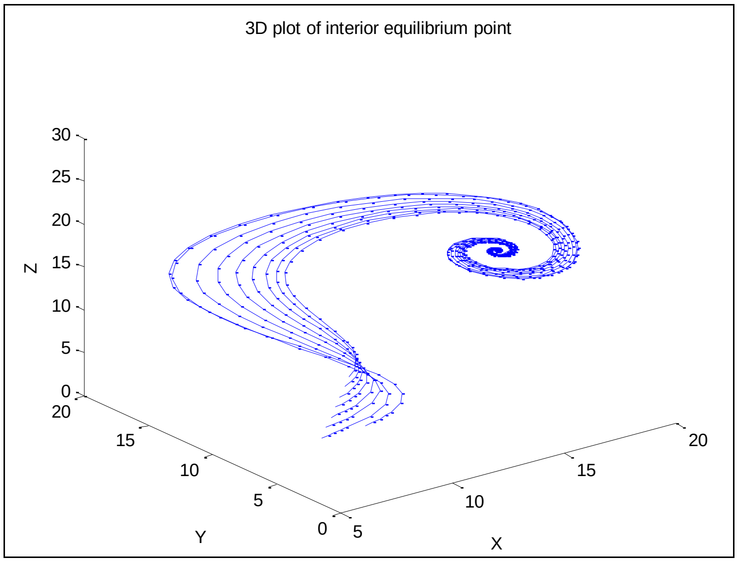

Fig 3: This is the phase portrait of the interior equilibrium point of the model system taking the parametric values as r=0.75; k =20; a=0.04; b=0.03; m=0.002; γ=0.16. In this case we take different initial values but ultimate we obtained the desire result that the all species are in stable equilibrium.

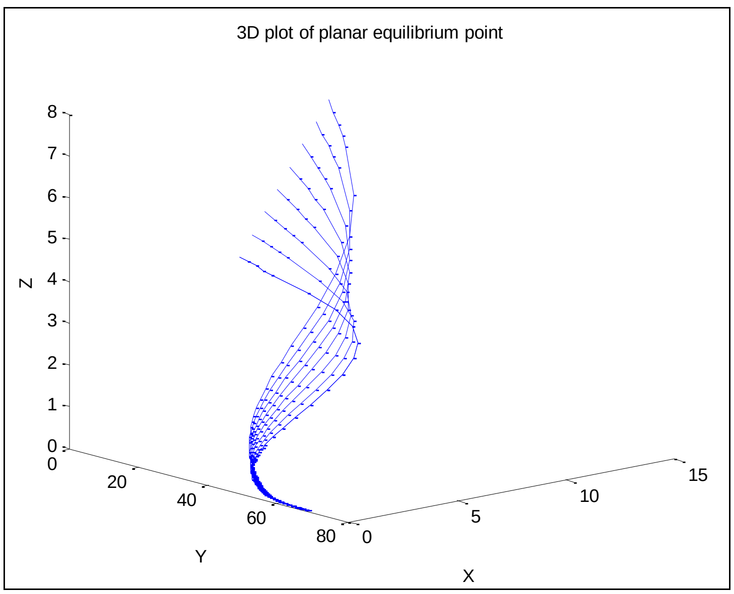

Fig 4: This is the phase portrait of the planar equilibrium point of the model system taking the parametric values as r=0.85; k =30; a=0.074; b=0.007; m=0.006; γ=0.66. In this case we take different initial values but ultimate we obtained the top predator free stable equilibrium.

5. Impact of harvesting phenomenon to reach MSY:

5.1 Prey harvesting at MSY level:

It is true that in one prey one predator system, the harvesting to reach at MSY level of prey species causes the extinction of predator species. The influence of prey harvesting in a food chain model consisting of more than two tropic levels has not been taken much attention by the researchers. We now considered the model system (1) and study the consequences of prey exploitation at MSY level. In all step here we always consider the harvesting function of the form $h = ex$. The system (1) with prey harvesting has the following equilibrium point.

First we consider the case when both predator free equilibrium point $E_1^*(ke^{\frac{\varepsilon}{r}}, 0, 0)$ exists if effort is smaller than the biotic potential of the prey. It is observe that prey biomass gradually decreases as effort increases.

Second we take that the top predator free equilibrium point is $E_2\left(\frac{m}{a}, \frac{r}{a}\log\left(\frac{ka}{m}\right) + \frac{\varepsilon}{a}, 0\right)$. In this case prey biomass remains at constant level and the intermediate predator biomass gradually increases but the top-predator biomass gradually decreases and ultimate goes to extinction from ecosystem if the effort is sufficiently large.

Coexisting equilibrium point is $E_3\left(ke^{\frac{-(be+a\gamma)}{br}}, \frac{\gamma}{b}, \frac{1}{b}ake^{\frac{-(a\gamma+be)}{br}} - \frac{m}{b}\right)$. In this case it is clear that prey and top predator biomass at equilibrium point decreases with increasing effort, but biomass of the middle predator at this equilibrium point remains at constant level still top predator is not collapsed from the ecosystem. Our aim is to invest the possible situation of extinction and persistence of the higher tropic levels while bottom topic level is exploited at MSY level. The yield at equilibrium is given by $Y(e) = eke^{\frac{-(be+a\gamma)}{br}}$. The yield curve has unique maximum when effort is taken as $e_{MSY} = eke^{\frac{-(be+a\gamma)}{b}}$.

5.2 Harvesting of the predator Y at MSY level

In this section we investigate the possible impacts of harvesting the predator species. We suppose that the predator (middle predator) is harvested according to the law of $h = ey$. Then the equilibrium biomass of the predator becomes $y_* = \frac{\gamma}{b}$ and the harvesting biomass is given by $Y = ey_*$. It is found that the equilibrium biomass of the predator is independent of effort, but the yield is linearly related to it. Also it is easy to check that the biomass of prey and top predator is functions of effort. Thus increasing effort produces increasing yield and reduces the prey and the top predator abundance, but biomass of the predator remains at constant level. Hence the top predator goes to extinction for some critical effort and we cannot pursue the MSY form the predator.

5.3 Harvesting of the top predator at MSY level

We now consider the top predator as the target species for harvesting in the model system (1). Then the biomass of all the species at equilibrium is functions effort. Lower employed effort produces smaller yield even in the presence of higher stock and higher effort gives also smaller yield due to the presence of lower stock of the top predator. Hence there must exist an intermediate value of the effort at which MSY can be obtained and the other two tropic levels survive.

6. Conclusion

In this paper we have focused the stability and coexistence scenario of species under harvesting aiming the maximum yield from ecosystem. We have tried to find out the stability analysis analytically and numerical support has been given by the Fig.1 and Fig.2. From fig. 3 and fig.4 shows the 3D phase portrait of interior and planar equilibrium point of the model system (1). This model is based on real world consideration but real world data are not available to us. So we consider numerous scenarios of biological feasible parameter value for this model system and tested different situation for different case. In this paper we considered and discussed the case to obtain maximum sustainable yield from both the prey and predator population. We have also investigated the impact of harvesting the prey and predator population at MSY level. Analytically it is observed that the prey exploitation at MSY level is a suitable harvesting policy towards the sustainable development and species conservation view points. We try to attention about the predator is harvested at MSY level, then the top predator must be driven to extinction. In this work we tried to develop suitable management tool to protect the entire ecosystem when species is exploited at MSY level of the model system.

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