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A property of the central automorphism with order of the centre is p

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Abstract
Let G be a group. Aut (G) denote the full automorphisms group of G. We discuss central automorphism with the property that if σ be the central automorphism than $(\sigma(x))^p = x^p$ When $|Z(G)| = p$, p is prime.

Keywords: central automorphism, centre, nilpotency class

1. Introduction
Let G be a group and G' and Z(G) denote the commutator subgroup and the centre of G respectively. Aut(G) is the automorphism group of G. An automorphism σ of G is called a class-preserving automorphism. If for each x ε G, ∃ an element t_x ε G st $\sigma(x) = t_x^{-1}$ x t_x. An automorphism f is called inner automorphism of for each x ε G, ∃ a fixed element g ε G s.t. $f(x) = g^{-1}$ x g [8]. Inn (G) is the set of all inner automorphisms of G. Whereas $Aut_c(G)$ denote the group of class-preserving automorphisms of G. An automorphism which is not inner is called outer automorphism. An automorphism σ of G is called a central automorphism if it commutes with all the inner automorphisms or equivalently, $x^{-1}\sigma(x)$ ε Z(G) ∀ x ε G. $Aut_Z(G)$ denote the group of all central automorphisms of G and let Aut_Z^z (G) denote the group of all those central automorphisms which fix the centre of G elementwise. Researchers have done valuable work in the field of automorphisms. More work yet to be done in this field [8].

2. Some basic facts of group [5, 8, 9]
A p-group G is regular If for x, y ε G, there is an element t ε H' s.t. H = <x, y>, such that $x^p y^p = (xy)^p t^p$, [5] A finite collection of normal subgroups H_i of a group G is a normal series for G if, $1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_r = G$. This normal series is a central series if $H_i/H_{i-1} \subseteq Z(G/H_{i-1})$ for $1 \leq i \leq r$. A group G is nilpotent if it has a central series. Subgroups and factor groups of nilpotent groups are nilpotent.
Given any group G, we define a central series as follows. Let $Z_0 = 1$ and $Z_1 = Z(G)$. The second center Z_2 is defined to be the unique subgroup such that $Z_2/Z_1 = Z(G/Z_1)$. We continue like this, inductively defining Z_n for $n > 0$ so that $Z_n/Z_{n-1} = Z(G/Z_{n-1})$.
The chain of normal subgroups.
$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots$
Constructed in this way is called upper central series of G. The upper central series may not actually be a central series for G because it may happen that $Z_i < G$ for all i. If $Z_r = G$ for some integer r, then $\{Z_i \mid 0 \leq i \leq r\}$ is a true central series and G is nilpotent.
If G is an arbitrary nilpotent group, then G is a term of its upper central series. As $G = Z_r$ for some integer $r \geq 0$, and the smallest integer r for which this happens is called the nilpotence class of G. Non trivial abelian groups have nilpotence class 1, and for non-abelian groups of nilpotency class 2 have quotient group G/Z(G) abelian.
[8, 9] An automorphism φ of G is called a class-preserving automorphism, if for each x ε G, ∃ an element g_x ε G s.t. $\varphi(x) = g_x^{-1}$ x g_x; and is called an inner automorphism of for each x ε G, ∃ a fixed element g ε G s.t. $\varphi(x) = g^{-1}xg$. An automorphism of a group is called an outer automorphism if it is not an inner automorphism. An automorphism φ of G is called a central automorphism If it commutes with all inner automorphisms or equivalently, if $g^{-1}\varphi(g)$ ε Z (G) ∀ g ε G.
The commutator of a, b ε G is $[a, b] = a^{-1}b^{-1}ab$ and the commutator subgroup G' of G is the subgroup of G generated by all commutators of G. A maximal subgroup of G is a proper subgroup S s.t. there is no subgroup H of G s.t. $S \subset H \subset G$. The Frattini subgroup φ(G) of G is the intersection of all maximal subgroups of G. If G has no maximal subgroup then φ(G) = G. An element a ε G is called non-generator of G, If G = <a, Y> then G = <Y>. φ(G) is exactly the set of all non-generators of G. Let G be a finite group then G is nilpotent if and only of G/φ(G) is nilpotent.
A cyclic group is generated by a single element. We denote a cyclic group of order n by C_n. The rank of a group is denoted by d(G), which is the smallest generating set of G. The least common multiple of orders of the elements of a finite group G is called the exponent of G.

2.2 Lemma
[5] Let G be finite. Then the following are equivalent.
1. G is nilpotent
2. Every nontrivial homomorphic image of G has a nontrivial center.
3. G appears as a member of its upper central series.

2.3 Theorem
[5] Let G be a (not necessarily finite) nilpotent group with

central series.
$1 = 1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq .... \subseteq H_r = G$, and let $1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq ....$ be the upper central series of G. Then $H_i \subseteq Z_i$ for $0 \leq i \leq r$, and in particular, $Z_r = G$.

Theorem 2.4: Let H is a subgroup of G, where G is a nilpotent group. Then $N_G(H) > H$ [5].

Theorem 2.5: For any group G, Inn(G) is a normal subgroup of Aut(G). Further Inn(G) ≅ G/Z(G), where Z(G) denotes the centre of G.

3. Results

Theorem 3.1: If G is a regular p-group, then for all x, y ε G and i, j ≥ 0 [8, 3].

$$[x^{p^i}, y^{p^j}] = 1 \text{ if and only if } [x, y]^{p^{i+j}} = 1$$

Theorem 3.2: If G is a purely non-abelian finite group, then $|Aut_z(G)| = |Hom(G/G', Z(G))|$ [8, 2]

Theorem 3.3: Let G be a non-abelian p-group of order p^4. Then one of the following holds [8, 7].
1. $|Z(G)| = p^2$, $|G'| = p$ and $G' \leq Z(G)$, and
2. $|Z(G)| = p$, $|G'| = p^2$ and $Z(G) \leq G'$

Theorem 3.4: Let G be a finite non-abelian p-group such that $Aut_z(G) = Z(Inn(G))$, Then $Z(G) \leq G']$ and Z(Inn(G)) is not cyclic [8, 4].

Theorem 3.5: Let G be a finite nilpotent group of class 2. Then exp (G') = exp (G/Z(G)), and in the decomposition of G/Z(G) into a direct product of cyclic groups, at least two factors of maximal order must occur [8, 6].

4. Property of the central automorphism with the order of the centre is p.

Let σ be the central automorphism of a group G. Let |Z(G)| = p.
We know that $x^{-1} \sigma(x) \in Z(G)$. It means that $o(x^{-1}\sigma(x)) = p$. We take the case when p = 3.
Therefore $(x^{-1} \sigma(x)) (x^{-1} \sigma(x)) (x^{-1} \sigma(x)) = e$, where e is the identity of G.
$(\sigma(x) x^{-1}) (x^{-1} \sigma(x)) (x^{-1} \sigma(x)) = e$
$\sigma(x) (x^{-1} \sigma(x)) x^{-1} (x^{-1} \sigma(x)) = e$
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$\sigma(x) \sigma(x) \sigma(x) x^{-1} x^{-1} x^{-1} = e$
This means $(\sigma(x))^3 = x^3$
Continuing like this we arrived at $(\sigma(x))^p = x^p$, where p is a prime. This property is open for discussion.

5. References

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