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Orbit of an eventually periodic points and periodic points

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Abstract
In this paper, Orbits of an Eventually periodic points and Periodic points were studied. The concept of an eventuality of an eventually periodic point introduced which helps in obtaining its orbit precisely. Unlike the case of a periodic point, the iterated images of an eventually periodic point need not always be periodic. Relation between orbit of eventually periodic points and orbit of a function were also obtained. Some results of considerable importance about the orbit of a point and relation with eventually periodic point were also proved. The equivalence was obtained between different points of a periodic function. Some results of the set theory that play an important role in our studies were included.
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1. Introduction
The concept of chaos dates back to 1880`s i.e. to the days of Poincare. More serious efforts have been done especially in the last about 50 years. Some new studies have been developed to help take care of chaos with mathematical precision or with as much of mathematical precision as is possible. By making use of knowledge of some mathematical concepts like dynamical systems and chaotic dynamical systems, vague or chaotic things can be changed into 'precise' things. There have been developed ways of studies where chaos can be viewed precisely. It took many years to come to a definition of chaos which can deliver the things. The first accepted definition of chaos is given by Devaney in [1]. It has been used by many e.g. see [2, 3, 5]. The study of chaos involves the concept of a dynamical system (X, f), where X is a topological space or a metric space, and f is a continuous self map on X. Then we have the concept of chaotic map or chaotic dynamical system or chaos. One of the initial part of the study of dynamical systems is study of self maps on the underlying set X of a dynamical system, at times without involving the topology or metric of X. There are concepts of periodic, eventually periodic, orbits etc for points of the domain of the self map. These and many other concepts (see e.g. [4, 6] like dense orbits, denseness of periodic points, self maps being topologically transitive and sensitive to initial conditions etc have many roles to play in the study of dynamical systems and chaotic dynamical systems. Orbit of a point in Dynamical Systems [11] helps in studying the relation between periodic and eventually periodic points of a function.
In this paper we find new observations about periodic and eventually points and their orbits which can be of help in the study of chaos. The results obtained are used to arrive at some results, which include results on chaotic maps, maps sensitive to initial conditions and sensitivity constants.

2. Notation and Definitions
Let X is a non-empty set. Let M(X) be the collection of all self maps on X, i.e. $M(X) = \{f \mid f: X \to X\}$. Let $f \in M(X)$. We define f^0 = Identity on X, $f^1 = f$, and for $n \in IN$, $f^{n+1} = fof^n$. A point $x \in X$ is called eventually periodic if for some nonnegative integer t, $f^t(x)$ is periodic. Let $x \in X$ be eventually periodic with eventual period k. Let t be the smallest nonnegative integer such that $f^t(x)$ is periodic with period k. Then we say that t is eventuality of x.
Let $f: X \to X$. Let $x \in X$. The set $\{f^n(x) \mid n \geq 0\}$ is called the orbit of x. The orbit of x is denoted by Orb(x) or Orb(x, f). A point $x \in X$ is called periodic if $f^k(x) = x$ for some $k \in IN$. The smallest k such that $f^k(x) = x$ is called the period of x. Let Per(f) be the set of all periodic points of X. By Orb(X) we shall denote the union of the orbits of all periodic points of X, i.e. $Orb(X) = \cup\{Orb(x): x \in Per(f)\}$. Let X is a topological space. Let $f: X \longrightarrow X$. The pair (X, f) is called a dynamical system if f is continuous.
A map $f: X\to X$ is called topologically transitive (TT) if for every pair of non empty open sets G, V in X, there exists some $m \in IN$ such that $f^m(G) \cap V \neq \varphi$ or equivalently $G\cap(f^m)^{-1}(V) \neq \varphi$. If f^n is transitive for each $n \in N$, then f is called totally transitive.
Let (X,f) be a dynamical system where X is a metric space. f is said to be sensitive two initial conditions (SIC) there exists a $\delta > 0$ such that for a given $x \in X$ and a neighborhood N(x) of

x, there exists some $y \in N(x)$ and some $v \in IN$ such that $d(f^v(x),f^v(y)) \geq \delta$; such δ is called a sensitivity constant.

3. Preliminaries

We shall need the following Remark. We may use it without mentioning it.

Remark 1.1. (i) Let $n \in IN$, then $f^{n+1} = f^n o f$. (ii) Let m, $n \in IN$, then $f^{n+m} = f^n o f^m$.

Remark 1.2. Let $x \in X$ be periodic with period k. Then $Orb(x) = \{x, f(x),\dots f^{k-1}(x)\}$.

Proof. For some $n \in IN$, $f^n(x) = f^r(x)$, for some integer r with $0 \leq r < k$. Therefore $f^n(x) \in \{x, f(x),\dots f^{k-1}(x)\}$ as $r < k$. Hence $Orb(x) = \{x, f(x),\dots f^{k-1}(x)\}$.

Lemma 1.3. Let $x \in X$. For an integer $t \geq 0$, let $f^t(x)$ be periodic with period k. Then, for $n \in IN$, $f^n(x) \in \{f^0(x), f^1(x), f^2(x),\dots f^{t+k-1}(x)\}$.

Proof. If $t = 0$, then x is periodic with period k. The result follows by Remark 1.2, as $f^n(x) = f^r(x)$, for some integer r with $0 \leq r < k$. Let $t \geq 1$. If $n < k+t$, then $n \leq t+k-1$. Therefore, $f^n(x) \in \{f^0(x), f^1(x), f^2(x),\dots f^{t+k-1}(x)\}$. If $n \geq k+t$, we write $n-t = mk + r$ for some m in IN and an integer r with $0 \leq r < k$. We have $f^n(x) = f^{mk+t+r}(x) = f^{t+r}(x)$. So $f^n(x) \in \{f^0(x), f^1(x), f^2(x),\dots f^{t+k-1}(x)\}$, as $t+r \leq k+t-1$.

Remark 1.4. Let $g: X \rightarrow X$. For G, $V \subset X$, $g(G) \cap V \neq \varphi$ iff $G \cap g^{-1}(V) \neq \varphi$.

Proof. Let $z \in g(G) \cap V$. Then there exists $x \in G$ such that $g(x) = z$, and $z \in V$. So $x \in G \cap g^{-1}(V)$. Let $z \in G \cap g^{-1}(V)$. Then $z \in G$ and $g(z) \in V$. Since $g(z) \in g(G)$, $g(G) \cap V \neq \varphi$.

4. Orbit of an Eventually periodic points

Proposition 2.1. Let $x \in X$ be periodic with period k. Then Orb(x) contains exactly k elements.

Proof. By Remark1.2, $Orb(x) = \{x, f(x),\dots f^{k-1}(x)\}$. Let $f^s(x)$, $f^r(x) \in \{x, f(x),\dots f^{k-1}(x)\}$ be such that $f^s(x) = f^r(x)$. Hence $\{x, f(x),\dots f^{k-1}(x)\}$ contains exactly k elements.

Proposition 2.2. Let $x \in X$ be periodic with period k.

(i) For an integer r with $0 \leq r < k$, $f^r(x)$ is periodic with period k and $Orb(f^r(x)) = Orb(x)$.

(ii) For $n \in IN$, $f^n(x)$ is periodic with period k and $Orb(f^n(x)) = Orb(x)$.

Proof. (i) $f^r(x) = f^r(f^k(x)) = f^k(f^r(x))$. So $f^r(x)$ is periodic. Let m be the period of $f^r(x)$. Now $k = mq$ for some $q \in IN$. $f^m(f^r(x)) = f^r(x)$, so $f^{m+r}(x) = f^r(x)$. Therefore, $f^{m+k}(x) = f^{m+r+k-r}(x) = f^{r+k-r}(x) = f^k(x) = x$. So $f^{m+k}(x) = x$. Thus $m+k = kq^*$ for some $q^* \in IN$. So $m = k(q^*-1)$. Since $k = mq$, we have $k = m$. To prove that $Orb(f^r(x)) = Orb(x)$, let $y = f^r(x)$. For $n \in IN$, $f^n(y) = f^{n+r}(x)$. Since $f^{n+r}(x) \in Orb(x)$, $f^n(y) \in Orb(x)$. So $Orb(f^r(x)) \subset Orb(x)$. For $f^s(x) \in Orb(x)$, $0 \leq s < k$. If $r \leq s$, then $f^s(x) = f^{s-r}(f^r(x)) = f^{s-r}(y)$. Therefore $f^s(x) \in Orb(f^r(x))$. Suppose $s < r$. $s+k-r > 0$ as $r < k$. $f^{s+k-r}(y) = f^{s+k-r}(f^r(x)) = f^{s+k}(x) = f^s(x)$. Since $f^{s+k-r}(y) \in Orb(y)$, $f^s(x) \in Orb(y)$.

(ii) $Orb(x) = \{x, f(x),\dots f^{k-1}(x)\}$. So $f^n(x) = f^r(x)$ for some integer r, with $0 \leq r < k$. Now the result follows from (i).

Corollary 2.3. Using Proposition 2.2(i) we have Per(f) = Orb(X).

Lemma 2.4. Let $x \in X$. For an integer $t \geq 0$, let $f^t(x)$ be periodic with period k. Then the following hold.

(i) For $t \leq s$, $f^s(x)$ is periodic with period k and $Orb(f^t(x)) = Orb(f^s(x))$.

(ii) For n, $m \in IN$, $f^{mk+nt}(x) = f^{nt}(x)$, in particular, $f^{m(k+t)}(x) = f^{mt}(x)$.

(iii) $f^{mk+t-r}(x) = f^{k+t-r}(x)$ for all $r \leq k$ and for all $m \geq 1$.

(iv) Let $m \in IN$. If $k < r \leq mk$, then $f^{mk+t-r}(x) = f^{k+t-s}(x)$ for some s, $0 \leq s < k$.

Proof. (i) Let $y = f^t(x)$. Since $t \leq s$, $s = t+r$ for some $r \geq 0$. $f^s(x) = f^{t+r}(x) = f^r(f^t(x)) = f^r(y)$. By Proposition 2.2(ii), $f^r(y)$ is periodic with period k, and $Orb(f^r(y)) = Orb(y)$. Therefore $f^s(x)$ is periodic with period k and $Orb(f^t(x)) = Orb(f^s(x))$.

(ii) Since $f^{mk+t}(x) = f^t(f^{mk}(x)) = f^t(x)$. Therefore, $f^{(n-1)t}(f^{mk+t}(x)) = f^{(n-1)t}(f^t(x))$. So $f^{mk+nt}(x) = f^{nt}(x)$. In particular taking, n = m, $f^{m(k+t)}(x) = f^{mt}(x)$.

(iii) Using induction on m, for m = 1, it is obvious. Suppose the result is true for m, i.e. $f^{mk+t-r}(x) = f^{k+t-r}(x)$. $f^{(m+1)k+t-r}(x) = f^{mk+t-r+k}(x) = f^k(f^{mk+t-r}(x)) = f^k(f^{k+t-r}(x)) = f^{2k+t-r}(x) = f^{k+k+t-r}(x) = f^k(f^{k+t-r}(x))$. Since $k+t-r \geq t$, $f^{k+t-r}(x)$ is periodic with period k, by (iii). Therefore $f^k(f^{k+t-r}(x)) = f^{k+t-r}(x)$.

(iv) We write $r = qk+s$ with $0 \leq s < k$. If $r = mk$, then s = 0 and m = q. In this case, we have $f^{mk+t-r}(x) = f^t(x) = f^{k+t}(x)$. If $r < mk$, then $(m-q) \geq 1$ and $mk-qk \geq k > s$. So $mk-qk-s>0$. Therefore $f^{mk+t-r}(x) = f^{mk+t-(qk+s)}(x) = f^{(m-q)k+t-s}(x) = f^{k+t-s}(x)$, using (iv), as $0 \leq s < k$ and $(m-q) \geq 1$.

Remark 2.5. Let $x \in X$ and $f^t(x)$ is periodic with period k. $f^s(x)$ is periodic with period k for each $s \geq t$. It is interesting to know about the totality of $f^s(x)$ which are periodic with period k. We get an answer to this question if we know about periodicity of $f^s(x)$ for $s < t$. We find an answer to the later in two parts. One of the parts is the following.

Proposition 2.6. For $x \in X$, let $t \geq 0$ and $f^t(x)$ is periodic with period k. For a nonnegative integer $s < t$, if $f^s(x)$ is periodic, then its period is k.

Proof. If $s > t$, then period of $f^s(x)$ is k by Lemma 2.4. Suppose $t > s$. Let ν be the period of $f^s(x)$. Since $s < t$, by Lemma 2.4, $f^t(x)$ is periodic with period ν. Hence $\nu = k$.

Remark 2.7. Let $x \in X$. Let t be a nonnegative integer. Suppose that $f^t(x)$ is periodic. Let k be the period of $f^t(x)$. Let s be a nonnegative integer. If $s > t$, by Lemma 2.4, $f^s(x)$ is periodic with period k. If $s < t$ and $f^s(x)$ is periodic, then, by Proposition 2.6, period of $f^s(x)$ is k. Using Lemma 2.4, it can be seen that there may exist $s < t$ such that $f^s(x)$ is periodic. Let $x \in X$ be eventually periodic. There exists a nonnegative integer t such that $f^t(x)$ is periodic. In view of Lemma 2.4 and Proposition 2.6, for every nonnegative integer s, the period of $f^s(x)$, whenever it is periodic, is k. We say that k is eventual period of an eventually periodic $x \in X$, if $f^t(x)$ is periodic with period k for any integer $t \geq 0$.

Remark 2.8. Let $x \in X$. If x is periodic with period k, then, the elements of $\{x, f(x),\dots f^{k-1}(x)\}$, which is Orb(x) are distinct. Suppose that x is eventually periodic with eventual period k. There exists an integer $t \geq 0$ such that $f^t(x)$ is periodic with period k. Since Orb (x) = $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$ for s > t using Lemma 2.4, $f^s(x)$ is periodic with period k and, therefore, $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$ = Orb(x) = $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$. We suppose that $k \geq 2$. For $0 \leq r \leq k-1$, let s = t+k-r. k+s-1 = 2k+t-(r+1). Since $r+1 \leq k$, by Lemma 2.4(v), $f^{2k+t-(r+1)}(x) = f^{k+t-(r+1)}(x)$. Thus $f^{2k+t-(r+1)}(x)$ and $f^{k+t-(r+1)}(x)$ are not distinct elements of $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$, where s = t+k-r. The elements of even $\{f^0(x), f^1(x),f^2(x),\ldots f^{k+t-1}(x)\}$ need not be distinct. If we take s < t, in view of Proposition 2.6, if $f^s(x)$ is periodic (which may be possible as mentioned in Remark 2.7), then its period is k. In this case also, the elements of $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+s-1}(x)\}$ need not be distinct.

Thus there is need to evolve a way so that we are sure that the elements of the orbit of an eventually periodic element $x \in X$ are distinct. For this we have the following definition. This will also settle the question of totality of $f^s(x)$ which are periodic with period k, as mentioned in Remark 2.5.

Remark 2.9. Let $x \in X$ be eventually periodic with eventual period k and eventuality t. Let s be a nonnegative integer. If $f^s(x) = f^t(x)$, then $f^s(x)$ is periodic with period k. Therefore, $t \leq s$.

The following results are possible using the concept of eventuality of an eventually periodic element.

Proposition 2.10. Let $x \in X$ be eventually periodic with eventual period k and eventuality t. Then, (i) For $0 \leq r, s < k+t$, if $f^s(x) = f^r(x)$, then s = r.

(ii) Let $0 \leq r < k \leq s$. If $f^s(x) = f^r(x)$, then s = r+kq for some nonnegative integer q.

(iii) For $\nu, s \in$ IN with $\nu \geq s \geq k$, if $f^s(x) = f^\nu(x)$, then $\nu = s+kq$ for some nonnegative integer q.

(iv) For ν, $s \in$ IN, if $f^s(x) = f^\nu(x)$, then $\nu = s+kq$ for some integer q.

Proof. Let $y = f^t(x)$. (i) We suppose that $r \leq s$. For s < k, if $f^r(x) = f^s(x)$ then r = s. Suppose that $k \leq s$. As s < k+t, k+t-s > o. Since $f^r(x) = f^s(x)$, $f^{r+k+t-s}(x) = f^{s+k+t-s}(x) = f^{k+t}(x) = f^t(x)$. Therefore, $t \leq r+ k+t-s$. So $0 \leq s-r \leq k$, as $r \leq s$. If s-r = k, then $f^k(f^r(x)) = f^{s-r}(f^r(x)) = f^{s-r+r}(x) = f^s(x) = f^r(x)$. Therefore, $f^r(x)$ is periodic. In view of Lemma 2.4 and Proposition 2.6, period of $f^r(x)$ is k. As t is eventuality of x, $t \leq r$. But r < t, as r+k = s < k+t. So s-r = k is not possible. Thus $0 \leq s-r < k$. Since $f^s(x) = f^r(x)$, $f^{s+t}(x) = f^{r+t}(x)$. So $f^s(y) = f^r(y)$. Either $0 \leq r < k \leq s$ or $k \leq r \leq s$. As y is periodic with k, s-r = qk for some nonnegative integer q. But s-r < k. Therefore q = 0. Hence s = r. (ii) Since $f^s(x) = f^r(x)$, so $f^s(y) = f^r(y)$. As y is periodic with k, $0 \leq r < k \leq s$, we have, s = r+kq for some nonnegative integer q. (iii) Since $f^s(x) = f^\nu(x)$, so $f^s(y) = f^\nu(y)$. As y is periodic with k and $\nu \geq s \geq k$, ν = s+ kq for some nonnegative integer q. (iv) follows from (i), (ii) and (iii).

Theorem 2.11. Let $x \in X$ be eventually periodic with eventual period k and eventuality t. Then Orb(x) contains exactly k+t elements.

Proof. Orb (x) = $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$. Let $f^s(x)$, $f^r(x) \in \{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$ be such that $f^s(x) = f^r(x)$. Since $0 < r, s < k+t$, by Proposition 2.10(i), s = r. Hence $\{f^0(x), f^1(x), f^2(x),\ldots f^{k+t-1}(x)\}$ contains exactly k+t elements.

Proposition 2.12. Let $x \in X$ be eventually periodic with eventual period k and eventuality t. For $0 < s < t$, $f^s(x)$ is not periodic.

Proof. Suppose $f^s(x)$ is periodic with period ν. Since $0 < s < t$, by Proposition 2.10, $\nu = k$. Thus $f^s(x)$ is periodic with period k. Therefore, by definition of eventuality of x, $t \leq s$. The contradiction arrived at proves that $f^s(x)$ is not periodic.

Remark 2.13. Let $x \in X$ be eventually periodic with eventual period k. Let ℓ be the eventuality of x. Now we have a clear position of some of the statements made in Remarks 2.5 and 2.7, namely about the totality of $f^s(x)$ which are periodic with period k and there may exist s < t such that $f^s(x)$ is periodic. In view of Lemma 2.4 and Proposition 2.10, $f^s(x)$ is periodic iff $s \geq \ell$, and then period of $f^s(x)$ is k. Thus $f^s(x)$ is not periodic iff $s < \ell$.

5. Orbits of periodic points

Lemma 3.1. Let x, $y \in X$ be periodic with periods k and m respectively.

(i) $f^\nu(y) = f^s(x)$ for some nonnegative integers ν and s, then k = m.

(ii) If $k \neq m$, then Orb(x) $\cap$ Orb(y) = φ.

Proof. By Proposition 2.10, $f^s(x)$ is periodic with period k and $f^\nu(y)$ is periodic with period m. Therefore k = m.

Theorem 3.2. Let x, $y \in X$ be periodic. Then either Orb(x) = Orb(y) or Orb(x) $\cap$ Orb(y) = φ.

Proof. Suppose that Orb(x) $\cap$ Orb(y) $\neq \varphi$. Then $f^\nu(y) = f^s(x)$ for some nonnegative integers ν and s. By Proposition 2.2(ii), Orb($f^s(x)$) = Orb(x) and Orb($f^\nu(y)$) = Orb(y). Therefore, Orb(x) = Orb(y).

If $x \in X$ is periodic, then $f^n(x)$ is periodic for every $n \in$ IN, but if $f^n(x)$ is periodic, the x need not be periodic.

Corollary 3.3. Let $x \in X$ be periodic with period k. Let $y \in X$ and s is a nonnegative integer. If $f^s(y)$ is periodic, then either Orb(x) = Orb($f^s(y)$) or Orb(x) $\cap$ Orb($f^s(y)$) = φ.

Proof. Let $z = f^s(y)$. x and z are periodic. The result follows by Theorem 3.2.

Remark 3.4. The above result need not be true if $x \in X$ is eventually periodic, but is not periodic. Then there exists $t \in$ IN such $f^t(x)$ is periodic. Let k be the period of $f^t(x)$. Let $y = f^t(x)$. In view of Proposition 2.2, Orb(y) = $\{y, f(y),\ldots f^{k-1}(y)\}$. Since Orb(x) = $\{f^0(x), f^1(x),f^2(x),\ldots f^{t+k-1}(x)\}$. Thus Orb(y) $\subset$ Orb(x). Since x is not periodic and by Proposition 2.2 every element of Orb(y) is periodic, $x \notin$ Orb(y). Therefore Orb(x) $\neq$ Orb(y).

Lemma 3.5. Let $x \in X$ be periodic with period k. Let $y \in X$ and s is a nonnegative integer. If $f^s(y) = f^r(x)$ for some nonnegative integer r, then $f^s(y)$ is periodic with period k and Orb($f^s(y)$) = Orb(x).

Proof. By Proposition 2.2(i), $f^r(x)$ is periodic with period k and

Orb(f^r(x)) = Orb(x), therefore f^s(y) is periodic with period k and Orb(f^s(y)) = Orb(x).

Remark 3.6. By Theorem 3.2. and Corollary 3.3, orbits of two elements of Per(f) are either equal or disjoint. If, for every x $\notin$ Per(f), we associate {x}, we get a partition of X. This partition gives an equivalence relation on X. But in view of Remark 3.4, we cannot include eventually periodic elements of X to get a partition of X.

Proposition 3.7. The following are equivalent.
(i) Orb(X) = Orb(z) for some z $\in$ Per(f).
(ii) Orb(X) = Orb(x) for every x $\in$ Per(f).
(iii) Orb(x) $\cap$ Orb(y) $\neq \varphi$ for every pair of x, y $\in$ Per(f).
Proof. (i) implies (ii). Let x $\in$ Per(f). Since Orb(x) $\subset$ Orb(X), by (i) Orb(x) $\subset$ Orb(z). Now by Theorem 3.2, Orb(x) = Orb(z). This proves (ii).
(ii) implies (iii). It follows as, by (ii), for x, y $\in$ Per(f), Orb(x) = Orb(y).
(iii) implies (i). Let x $\in$ Per(f). In view of (iii) and Theorem 3.2, Orb(x) = Orb(y) for every y $\in$ P_R(X). Therefore (ii) and (i) hold.

Proposition 3.8. If Orb(X) is infinite, then (i) for given x $\in$ Per(f), there exists y $\in$ Per(f) such that Orb(x) $\cap$ Orb(y) = φ. (ii) there exist x, y $\in$ Per(f) such that Orb(x) $\cap$ Orb(y) = φ.
Proof. (i) Suppose there exist z $\in$ Per(f) such that Orb(z) $\cap$ Orb(y) $\neq \varphi$ for every y $\in$ Per(f). Then by Theorem 3.2, Orb(z) = Orb(y) for every y $\in$ Per(f). Thus Orb(X) = Orb(z) which is not possible as Orb(z) is finite. (ii) follows from (i).

Proposition 3.9. Let X be infinite. If Per(f) is dense in X, then (i) for given x $\in$ Per(f) there exists y $\in$ Per(f) such that Orb(x) $\cap$ Orb(y) = φ.(ii) there exist x, y $\in$ Per(f) such that Orb(x) $\cap$ Orb(y) = φ.
Proof. By Corollary 2.3, Per(f) = Orb(X). Since Per(f) is dense, Orb(X) is infinite. Now the result follows by Proposition 3.8.

Proposition 3.10. Let x and y be two periodic elements of X with periods k and m respectly such that y $\in$ X – Orb(p). Let δ^{min} = inf{d(f^r(x)), d(f^s(y)) | $0 \leq r \leq k-1$, $0 \leq s \leq m-1$}. Let z $\in$ X. Then (a) $\delta^{min} \leq$ d(z,Orb(x)) + d(z,Orb(y)). (b) either d(z,Orb(x)) $\geq \delta^{min}/2$ or d(z,Orb(y)) $\geq \delta^{min}/2$. (c) If z $\in$ Orb(y), then $\delta^{min} \leq$ d(z,Orb(x)).
Proof. $\delta^{min} > 0$ as by Theorem 3.2, Orb(x) $\cap$ Orb(y) = φ. There exit some i and j, with $0 \leq i \leq k-1$ and $0 \leq j \leq m-1$, such that d(z,Orb(x)) = d(z,f^i(x)) and d(z,Orb(y)) = d(z,f^j(y)). (a)We have $\delta^{min} \leq$ d(f^i(x)), d(f^j(y)) $\leq$ d(z,f^i(x)) + d(z,f^j(y)). So $\delta^{min} \leq$ d(z,Orb(x)) + d(z,Orb(y)). (b) Because of (a) it is not possible that d(z,Orb(x)) < $\delta^{min}/2$ and d(z,Orb(y)) < $\delta^{min}/2$. Therefore (b) holds. (c) If z $\in$ Orb(y), then d(z,Orb(y)) = 0. Now by (a) $\delta^{min} \leq$ d(z,Orb(x)).

6. Conclusion

In Section 1, there are definitions, notation and preliminaries. For completeness sake, we have included some standard definitions and some preliminary results of Set theory. In Section 2, which include orbit of an eventually periodic points were studied. We also proved some results based on eventually periodic points. Unlike the case of a periodic point, the iterated images of an eventually periodic point need not always be periodic. Using eventuality of eventually periodic point complete information about periodicity of iterated images of an eventually periodic point and their period has been obtained. We also proved the necessary and sufficient condition for the periodicity of any function with the eventuality of a point of that function. In Section 3, this includes orbit of periodic points, starts with relations of different periodic points with regard to denseness of the set of all periodic points. In this section we also proved the orbits of two points of a function are either equal or disjoint. We also showed the equivalence between the orbit of a point and the set of a certain function.

References

1. Banks J, Brooks J, Cairns G, Davis G, Stacey P. On Devaney`s Definition of Chaos, The American Mathematical Monthly. 1992; 99(4):332-334.
2. Barnsely M. Fractals Everywhere, Acad. Press, 1988.
3. Birkhoff GD. Dynamical Systems, AMS Colloq. Publ., Collected mathematical AMS. 1950-1927; 9:3.
4. Frink O. Topology in Lattices, Trans. Amer. Math. Soc, 1942, 569-582.
5. Anima Nagar and Puneet Sharma, On dynamics of Circle Map.
6. Whyburn GT. Analytic Topology, AMS Colloq. Publ, 1942, 28.
7. Ernest Micheal, Topologies on spaces of subsets.
8. Devaney RL. Chaotic Dynamical Systems, by Add. Wesley, 1987.
9. Eckmann JP, Ruelle D. Ergotic Theory of Chaos and strange Attracters, Rev. Mod. Phy, 1985, 57.
10. Gottschalk, and Hedlund. Topological Dynamics, AMS Colloq. Publ, 1955, 36.
11. Babu Lal, Aseem Miglani, Vinod Kumar. Orbit of a point in Dynamical Systms, Mathematical Journal of Interdisciplinary sciences. 2016; 4(2):141-149.